CCFU Proof 4

$\mathrm{sig}(G_4) = (3, 1)$

Given. $\mathrm{spec}(A_4) = \{\varphi,\ 1/\varphi,\ +1,\ -1\}$.

This spectrum is derived from C_2 by separating $\mathrm{spec}(A_2) = \{\varphi,\ -1/\varphi\}$ into signs and magnitudes.
[Dependency: Theory #15c / CCFU Paper I]

Construction. Define $\tilde{G}$ in the eigenbasis by the reciprocal pairing rule:

$$\tilde{G}_{ij} = \begin{cases} 1 & \text{if } \lambda_i \cdot \lambda_j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Reciprocal pair $(\varphi,\ 1/\varphi)$: $\varphi \cdot (1/\varphi) = 1$, giving the off-diagonal block $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with eigenvalues $+1, -1$. Contribution to signature: $(1, 1)$.

Self-reciprocal modes $(+1$ and $-1)$:

$$(+1) \cdot (+1) = 1 \ \rightarrow \ \text{diagonal entry } +1, \qquad (-1) \cdot (-1) = 1 \ \rightarrow \ \text{diagonal entry } +1.$$

Contribution to signature: $(2, 0)$.

Total signature:

$$(1, 1) + (2, 0) = (3, 1).$$

Therefore $\mathrm{sig}(G_4) = (3, 1)$. ■

Verification of invariance. In the eigenbasis, $D = \mathrm{diag}(\varphi,\ 1/\varphi,\ +1,\ -1)$. For every nonzero entry $\tilde{G}_{ij}$, we have $\lambda_i \lambda_j = 1$, hence

$$(D^\top \tilde{G}\, D)_{ij} = \lambda_i \lambda_j\, \tilde{G}_{ij} = \tilde{G}_{ij}.$$

All zero entries remain zero. Therefore $D^\top \tilde{G}\, D = \tilde{G}$.
Standard basis. If $A_4 = P\, D\, P^{-1}$, define $G_4 = P^{-\top} \tilde{G}\, P^{-1}$. Then:

$$A_4^\top G_4\, A_4 = (PDP^{-1})^\top (P^{-\top} \tilde{G}\, P^{-1})(PDP^{-1}) = P^{-\top} D^\top \tilde{G}\, D\, P^{-1} = P^{-\top} \tilde{G}\, P^{-1} = G_4. \qquad ■$$